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On the stability and triviality of scalar quantum electrodynamics

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Abstract. We study the existence of a stable ground state for scalar QED in four dimensions within a variational approach. The results are similar to those for $\lambda\phi^4$: in its regularised version there is a metastable ground state which becomes stable when the cut-off is removed. However, contrary to what happens for $\lambda\phi^4$ the bare coupling constant λ_B does not now have to be negative. The theory is interacting and the renormalised coupling constant is negative, $\lambda < 0$.

1. Introduction

Our simplest field theory in four dimensions, $\lambda\phi^4$, is very likely to be trivial when λ_B, the bare coupling constant, is positive ([1], footnote 8). This condition is needed for the convergence of the functional integrals, which otherwise would have to be regularised for large quantum fluctuations. Recently Stevenson [2] has found within a variational approach [3] that for $\lambda_B < 0$ but vanishing when the cut-off is removed, the theory has a stable ground state and is interacting with $\lambda < 0$. More precisely the theory has a metastable ground state in its regularised version but the lifetime of this state diverges when one removes the cut-off. He calls such a theory precarious. Notice that this result is telling us that the theory only exists, is finite, stable and interacting when it is asymptotically free, as happens for the 'wrong sign' $\lambda\phi^4$ theory. The inclusion of odd terms in the field does not alter these conclusions for ϕ^4 [4].

The aim of this work is to present a similar study for scalar electrodynamics. Scalar electrodynamics is our simplest gauge theory and thus an extremely interesting model to study as carefully as possible. We want to know, within the variational approach mentioned above, whether the theory exists, i.e. is stable and interacting, and if so, whether it shows spontaneous symmetry breakdown as predicted by the one-loop corrections [5]. It should be mentioned, however, that the variational ansatz we follow in this paper is very likely inadequate for the study of the broken phase and so not much can be said with confidence with respect to spontaneous symmetry breaking†.

With this qualification in mind our results are the following: the theory only exists for very special values of the two coupling constants, λ_B being vanishing when the cut-off is removed. However λ_B does not now have to be negative and this makes the study of this theory amenable to those methods which require $\lambda_B > 0$. The renormalised quartic coupling constant is negative, $\lambda < 0$. Furthermore there is no spontaneous symmetry breakdown and it looks as if the theory is asymptotically free.

† One of us (RT) thanks P M Stevenson for pointing this out to him.

Let us finally mention that the variational method we use is non-perturbative and, as shown by Stevenson [6], does not reproduce perturbation theory. We believe it to be an adequate method for the study of interaction and existence because it leads to an upper bound of the energy density. We do not consider here the critical comparison with the one-loop effective potential, which is a perturbative magnitude.

2. The variational analysis

The Hamiltonian density of scalar electrodynamics is

$$\mathscr{H} = \dot{\phi}^{\dagger}\dot{\phi} + \nabla\phi^{\dagger}\cdot\nabla\phi + m_{\mathrm{B}}^{2}\phi^{\dagger}\phi + \lambda_{\mathrm{B}}(\phi^{\dagger}\phi)^{2} + \mathrm{i}e_{\mathrm{B}}(\nabla\phi^{\dagger})\phi\boldsymbol{A} - \mathrm{i}e_{\mathrm{B}}\phi^{\dagger}(\nabla\phi)\cdot\boldsymbol{A} - e_{\mathrm{B}}^{2}\phi^{\dagger}\phi A^{\dagger}A_{\mu}$$
$$+\tfrac{1}{2}\dot{\boldsymbol{A}}\cdot\dot{\boldsymbol{A}} - \tfrac{1}{2}(\nabla A^{0})\cdot(\nabla A^{0}) + \tfrac{1}{2}(\nabla^{i}A^{j})(\nabla^{i}A^{j} - \nabla^{j}A^{i}) - \xi_{\mathrm{B}}[\dot{A}^{0}\dot{A}^{0} - (\nabla\boldsymbol{A})^{2}] \quad (1)$$

where the subscript B indicates that all parameters are bare and the last term is the gauge fixing term. Notice that m_{B}^{2} can be negative. The computation of the ground state energy within the Gaussian approximation is straightforward [2]. One writes

$$\phi(x) = \phi_{0} + \int \frac{\mathrm{d}^{3}k}{(2\pi)^{3}2\omega_{k}(\Omega^{2})}[b_{\Omega}(\boldsymbol{k})\exp(-\mathrm{i}kx) + d_{\Omega}^{\dagger}(\boldsymbol{k})\exp(\mathrm{i}kx)]$$
$$A^{\mu}(x) = \int \frac{\mathrm{d}^{3}q}{(2\pi)^{3}2\omega_{q}(\Delta^{2})}\sum_{s=0}^{3}[a_{\Delta}(\boldsymbol{q},s)\varepsilon_{\Delta}^{\mu}(\boldsymbol{q},s)\exp(-\mathrm{i}qx) + a_{\Delta}^{\dagger}(\boldsymbol{q},s)\varepsilon_{\Delta}^{\mu *}(\boldsymbol{q},s)\exp(\mathrm{i}qx)] \quad (2)$$

where

$$\omega_{k}(\Omega^{2}) = (\boldsymbol{k}^{2}+\Omega^{2})^{1/2} \qquad \omega_{q}(\Delta^{2}) = (\boldsymbol{q}^{2}+\Delta^{2})^{1/2}$$
$$[b_{\Omega}(\boldsymbol{k}), b_{\Omega}^{\dagger}(\boldsymbol{k}')] = [d_{\Omega}(\boldsymbol{k}), d_{\Omega}^{\dagger}(\boldsymbol{k}')] = (2\pi)^{3}2\omega_{k}(\Omega^{2})\delta(\boldsymbol{k}-\boldsymbol{k}')$$
$$[a_{\Delta}(\boldsymbol{q},s), a_{\Delta}^{\dagger}(\boldsymbol{q}'s')] = -(2\pi)^{3}2\omega_{q}(\Delta^{2})g_{ss'}\delta(\boldsymbol{q}-\boldsymbol{q}')$$
$$b_{\Omega}(\boldsymbol{k})|0_{\Omega\Delta}\rangle = d_{\Omega}(\boldsymbol{k})|0_{\Omega\Delta}\rangle = a_{\Delta}(\boldsymbol{q},s)|0_{\Omega\Delta}\rangle = 0$$
$$\langle 0_{\Omega\Delta}|0_{\Omega\Delta}\rangle = 1$$
$$\varepsilon^{\mu}(\boldsymbol{q},s)\varepsilon_{\mu}^{*}(\boldsymbol{q},s') = g_{ss'} \qquad \sum_{s} g_{ss}\varepsilon^{\mu}(\boldsymbol{q},s)\varepsilon^{\nu *}(\boldsymbol{q},s) = g^{\mu\nu} \quad (3)$$

and Ω and Δ are as yet undetermined functions of the constant background field ϕ_0. Then an upper bound of the ground state energy density is given by the minimum of

$$\mathscr{V}(\phi_{0};\Omega,\Delta)\Big|_{\substack{\Omega=\Omega(\phi_{0})\\ \Delta=\Delta(\phi_{0})}} \equiv \langle 0_{\Omega\Delta}|\mathscr{H}|0_{\Omega\Delta}\rangle\Big|_{\substack{\Omega=\Omega(\phi_{0})\\ \Delta=\Delta(\phi_{0})}} \quad (4)$$

when ϕ_0 is varied in the whole parameter space, $\Omega(\phi_0)$ and $\Delta(\phi_0)$ having been fixed before by minimisation. Equation (4) has ultraviolet divergences of two kinds; one group is absorbed by renormalisation of the mass and coupling constant of the scalar field and the rest correspond to the zero-point energy which we will just subtract.

The computation of (4) is simple and leads to

$$\langle 0_{\Omega\Delta}|\mathscr{H}|0_{\Omega\Delta}\rangle = 2I_{1}(\Omega^{2}) - \Omega^{2}I_{0}(\Omega^{2}) + m_{\mathrm{B}}^{2}\alpha + m_{\mathrm{B}}^{2}I_{0}(\Omega^{2}) + \lambda_{\mathrm{B}}\alpha^{2} + 4\lambda_{\mathrm{B}}I_{0}(\Omega^{2})\alpha + 2\lambda_{\mathrm{B}}I_{0}^{2}(\Omega^{2})$$
$$+2e_{\mathrm{B}}^{2}I_{0}(\Delta^{2})\alpha + 2e_{\mathrm{B}}^{2}I_{0}(\Omega^{2})I_{0}(\Delta^{2}) + (3+2\xi_{\mathrm{B}})(I_{1}(\Delta^{2}) - \tfrac{1}{2}\Delta^{2}I_{0}(\Delta^{2})) \quad (5)$$

where

$$\alpha \equiv \phi_0^* \phi_0 \geqslant 0$$

$$I_n(\Omega^2) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k(\Omega^2)} (\omega_k^2(\Omega^2))^n. \tag{6}$$

Notice that $I_n(\Omega^2)$ is divergent for $n \geqslant -1$. We will regularise the theory with a symmetric cut-off Λ. The domain of the validity of the regularised theory is

$$0 \leqslant \Omega^2, \Delta^2, \alpha \ll \Lambda^2. \tag{7}$$

Before going on let us recall here that the effective potential for a gauge theory is gauge dependent [7], as corresponds to its off-shell character, but that its value at stationary points is gauge independent [8], as corresponds to its interpretation as vacuum energy density. Thus our study of the existence, i.e. the boundedness from below, is gauge independent as can furthermore be checked in the following, in spite of the appearance of ξ_B in many intermediate formulae. For the same reason the analysis of spontaneous symmetry breaking is gauge independent.

Let us renormalise the theory at $\alpha = 0$. In order to obtain the renormalised parameters we will start by obtaining $\Omega_0^2 \equiv \Omega^2(0)$ and $\Delta_0^2 \equiv \Delta^2(0)$ by minimisation of $\mathcal{V}(0; \Omega_0^2, \Delta_0^2)$. From (5)

$$\begin{aligned}
\partial \mathcal{V}(0; \Omega_0^2, \Delta_0^2)/\partial \Omega_0^2 &= -\tfrac{1}{2} I_{-1}(\Omega_0^2)[m_B^2 - \Omega_0^2 + 4\lambda_B I_0(\Omega_0^2) + 2e_B^2 I_0(\Delta_0^2)] \\
\partial \mathcal{V}(0; \Omega_0^2, \Delta_0^2)/\partial \Delta_0^2 &= -\tfrac{1}{2} I_{-1}(\Delta_0^2)[2e_B^2 I_0(\Omega_0^2) - (\tfrac{3}{2} + \xi_B)\Delta_0^2]
\end{aligned} \tag{8}$$

which leads to

$$\begin{aligned}
& m_B^2 - \Omega_0^2 + 4\lambda_B I_0(\Omega_0^2) + 2e_B^2 I_0(\Delta_0^2) = 0 \\
& 2e_B^2 I_0(\Omega_0^2) - \Delta_0^2(\tfrac{3}{2} + \xi_B) = 0 \\
& \Omega_0^2, \Delta_0^2 \geqslant 0
\end{aligned} \tag{9}$$

or if (9) do not have a solution which is the absolute minimum of $\mathcal{V}(0; \Omega_0^2, \Delta_0^2)$, to either

$$\begin{aligned}
& \Omega_0^2 = 0 \\
& 2e_B^2 I_0(0) - \Delta_0^2(\tfrac{3}{2} + \xi_B) = 0 \\
& \Delta_0^2 \geqslant 0
\end{aligned} \tag{10}$$

or

$$\begin{aligned}
& \Delta_0^2 = 0 \\
& m_B^2 - \Omega_0^2 + 4\lambda_B I_0(\Omega_0^2) + 2e_B^2 I_0(0) = 0 \\
& \Omega_0^2 \geqslant 0
\end{aligned} \tag{11}$$

or finally

$$\Omega_0^2 = \Delta_0^2 = 0. \tag{12}$$

Let us consider each of these cases in turn.

2.1. $\Omega_0^2>0, \Delta_0^2>0$

The minimum is given by (9) and is in the interior of the Ω_0^2, Δ_0^2 space. For this to be possible necessarily

$$\xi_B+\tfrac{3}{2}>0. \tag{13}$$

This inequality is inherent to the approach we follow, in which the trial fields are free. Indeed $\xi_B+\frac{3}{2}$ is nothing but the coefficient of the trace of the free Euclidean gauge field propagator, which is required to be positive. Equation (13) is a requirement of the method, and has no other significance (this is analogous to the requirement $\xi_B>0$ of the one-loop effective potential calculation which otherwise becomes complex [7]).

In order to ensure that the stationary point is a minimum (and not a saddle point or maximum) one needs

$$1+2\lambda_B I_{-1}(\Omega_0^2)>0 \tag{14}$$

$$d\equiv 2I_0(\Omega_0^2)(1+2\lambda_B I_{-1}(\Omega_0^2))-e_B^2\Delta_0^2 I_{-1}(\Omega_0^2)I_{-1}(\Delta_0^2)>0. \tag{15}$$

One can check that these are not only sufficient but also necessary conditions for (9) to give a local minimum.

The renormalised mass is given by

$$m^2\equiv\left.\frac{\mathrm{d}\mathcal{V}(\alpha;\Omega^2,\Delta^2)}{\mathrm{d}\alpha}\right|_{\alpha=0}=m_B^2+4\lambda_B I_0(\Omega_0^2)+2e_B^2 I_0(\Delta_0^2) \tag{16}$$

which from the first of (9) implies

$$m^2=\Omega_0^2>0. \tag{17}$$

In this phase the renormalised mass squared defined at the origin is positive.

The renormalised coupling constant is given by

$$\begin{aligned}\lambda &\equiv \tfrac{1}{2}\mathrm{d}^2\mathcal{V}(\alpha;\Omega^2,\Delta^2)/\mathrm{d}\alpha^2|_{\alpha=0}\\ &=\lambda_B-2\lambda_B\Omega_0^{2'}I_{-1}(\Omega_0^2)+\tfrac{1}{4}(\Omega_0^{2'})^2 I_{-1}(\Omega_0^2)(1+2\lambda_B I_{-1}(\Omega_0^2))-e_B^2\Delta_0^{2'}I_{-1}(\Delta_0^2)\\ &\quad+\tfrac{1}{2}e_B^2\Omega_0^{2'}\Delta_0^{2'}I_{-1}(\Omega_0^2)I_{-1}(\Delta_0^2)+\tfrac{1}{2}e_B^2(I_0(\Omega_0^2)/\Delta_0^2)I_{-1}(\Delta_0^2)(\Delta_0^{2'})^2\end{aligned} \tag{18}$$

where

$$\Omega_0^{2'}\equiv \mathrm{d}\Omega^2/\mathrm{d}\alpha|_{\alpha=0} \qquad \Delta_0^{2'}\equiv \mathrm{d}\Delta^2/\mathrm{d}\alpha|_{\alpha=0}.$$

These parameters are fixed by minimisation of λ, which corresponds to minimisation of $\mathcal{V}(\alpha;\Omega^2(\alpha),\Delta^2(\alpha))$ for small α. This leads to

$$\begin{aligned}\Omega_0^{2'}&=(2/d)[4\lambda_B I_0(\Omega_0^2)-\Delta_0^2 e_B^2 I_{-1}(\Delta_0^2)]\\ \Delta_0^{2'}&=(2/d)\Delta_0^2\end{aligned} \tag{19}$$

which substituted into (18) gives

$$\lambda=\lambda_B-[8\lambda_B^2 I_{-1}(\Omega_0^2)I_0(\Omega_0^2)+e_B^2\Delta_0^2 I_{-1}(\Delta_0^2)(1-2\lambda_B I_{-1}(\Omega_0^2))]d^{-1} \tag{20}$$

The formula

$$\mathrm{d}I_n(\Omega^2)/\mathrm{d}\Omega^2=(n-\tfrac{1}{2})I_{n-1}(\Omega^2) \tag{21}$$

has been repeatedly used. We will furthermore need the expressions

$$I_1(\Omega^2)-I_1(\Omega_0^2)=\tfrac{1}{2}(\Omega^2-\Omega_0^2)I_0(\Omega_0^2)-\tfrac{1}{8}(\Omega^2-\Omega_0^2)^2I_{-1}(\Omega_0^2)+\Sigma(\Omega^2,\Omega_0^2)$$

$$\Sigma(\Omega^2,\Omega_0^2)=\frac{1}{128\pi^2}\left(2\Omega^4\ln\frac{\Omega^2}{\Omega_0^2}-(\Omega^2-\Omega_0^2)(3\Omega^2-\Omega_0^2)\right)+O\left(\frac{1}{\Lambda^2}\right)$$

$$I_0(\Omega^2)-I_0(\Omega_0^2)=-\tfrac{1}{2}(\Omega^2-\Omega_0^2)I_{-1}(\Omega_0^2)+\Gamma(\Omega^2,\Omega_0^2)$$

$$\Gamma(\Omega^2,\Omega_0^2)=\frac{1}{16\pi^2}\left(\Omega^2\ln\frac{\Omega^2}{\Omega_0^2}-\Omega^2+\Omega_0^2\right)+O\left(\frac{1}{\Lambda^2}\right)=2\frac{d\Sigma(\Omega^2,\Omega_0^2)}{d\Omega^2}\geqslant 0$$

$$I_{-1}(\Omega^2)-I_{-1}(\Omega_0^2)=-\frac{1}{8\pi^2}\ln\frac{\Omega^2}{\Omega_0^2}+O\left(\frac{1}{\Lambda^2}\right)=-2\frac{d\Gamma(\Omega^2,\Omega_0^2)}{d\Omega^2}$$

$$I_{-1}(\Omega^2)=\frac{1}{8\pi^2}\left(\ln\frac{4\Lambda^2}{\Omega^2}-2\right)+O\left(\frac{1}{\Lambda^2}\right). \tag{22}$$

We now subtract a UV divergent constant which contains the zero-point energy by considering

$$\mathscr{E}(\alpha;\Omega^2(\alpha),\Delta^2(\alpha))\equiv\mathscr{V}(\alpha;\Omega^2(\alpha),\Delta^2(a))-\mathscr{V}(0;\Omega_0^2,\Delta_0^2) \tag{23}$$

which, using (22), gives

$$\begin{aligned}\mathscr{E}(\alpha;\Omega^2,\Delta^2)=&\tfrac{1}{4}(\Omega^2-m^2)^2I_{-1}(m^2)-(\Omega^2-m^2)\Gamma(\Omega^2,m^2)+2\Sigma(\Omega^2,m^2)\\&+2\lambda_B[(\alpha-\tfrac{1}{2}(\Omega^2-m^2)I_{-1}(m^2)+\Gamma(\Omega^2,m^2)]^2\\&+m^2\alpha-\lambda_B\alpha^2+2e_B^2[\alpha-\tfrac{1}{2}(\Omega^2-m^2)I_{-1}(m^2)-\Gamma(\Omega^2,m^2)]\\&\times[-\tfrac{1}{2}(\Delta^2-\Delta_0^2)I_{-1}(\Delta_0^2)+\Gamma(\Delta_1^2\Delta_0^2)]\\&+2(e_B^2I_0(m^2)/\Delta_0^2)[\tfrac{1}{4}(\Delta^2-\Delta_0^2)^2I_{-1}(\Delta_0^2)-(\Delta^2-\Delta_0^2)\Gamma(\Delta^2,\Delta_0^2)+2\Sigma(\Delta^2,\Delta_0^2)]\end{aligned} \tag{24}$$

where $\Omega^2=\Omega^2(\alpha)$ and $\Delta^2=\Delta^2(\alpha)$ are the functions of α which minimise (24).

Notice that for fixed α and Ω^2 and Δ^2 large (24) leads to

$$\begin{aligned}\mathscr{E}(\alpha,\Omega^2,\Delta^2)\underset{\Omega^2,\Delta^2\,\text{large}}{\simeq}&\tfrac{1}{4}\Omega^4(1+2\lambda_BI_{-1}(m^2))I_{-1}(m^2)-\frac{1}{32\pi^2}\Omega^4\ln\frac{\Omega^2}{m^2}(1+4\lambda_BI_{-1}(m^2))\\&+\tfrac{1}{2}e_B^2\Omega^2\Delta^2I_{-1}(m^2)I_{-1}(\Delta_0^2)+\frac{e_B^2}{2\Delta_0^2}\Delta^4I_0(m^2)I_{-1}(\Delta_0^2)\end{aligned} \tag{25}$$

where we have kept a subdominant term for later convenience. From (14) this expression is positive and blows up. Thus $\Omega^2(\alpha)$ and $\Delta^2(\alpha)$ are either given by a stationary point of (24)

$$\begin{aligned}&m^2-\bar{\Omega}^2+4\lambda_B[\alpha-\tfrac{1}{2}(\bar{\Omega}^2-m^2)I_{-1}(m^2)+\Gamma(\bar{\Omega}^2,m^2)]\\&\qquad+2e_B^2[-\tfrac{1}{2}(\bar{\Delta}^2-\Delta_0^2)I_{-1}(\Delta_0^2)+\Gamma(\bar{\Delta}^2,\Delta_0^2)]=0\\&\alpha-\tfrac{1}{2}(\bar{\Omega}^2-m^2)I_{-1}(m^2)+\Gamma(\bar{\Omega}^2,m^2)-(I_0(m^2)/\Delta_0^2)(\bar{\Delta}^2-\Delta_0^2)=0\end{aligned} \tag{26}$$

or by

$$\begin{aligned}&\Omega^2=0\\&\alpha+\tfrac{1}{2}m^2I_{-1}(m^2)+m^2/16\pi^2-(I_0(m^2)/\Delta_0^2)(\bar{\Delta}^2-\Delta_0^2)=0\end{aligned} \tag{27}$$

or by

$$\Delta^2=0$$

$$m^2-\bar{\Omega}^2+4\lambda_B[\alpha-\tfrac{1}{2}(\bar{\Omega}^2-m^2)I_{-1}(m^2)+\Gamma(\bar{\Omega}^2, m^2)]$$
$$+2e_B^2(\tfrac{1}{2}\Delta_0^2 I_{-1}(\Delta_0^2)+\Delta_0^2/16\pi^2)=0 \tag{28}$$

or finally by

$$\Omega^2=\Delta^2=0 \tag{29}$$

depending on which gives the absolute minimum. The last two cases imply from (24) that

$$e_B^2\sim m^4/I_0(m^2)I_{-1}(\Delta_0^2)\Delta_0^2 \tag{30}$$

if the energy density is to be finite when $\Lambda\to\infty$, as is required by a correctly renormalised theory. It can then be seen, plugging (30) into (24), that minimisation leads to $\Delta^2=\Delta_0^2>0$ for all α, contrary to what (28) and (29) imply. Thus only (26) or (27) apply. Then

$$\bar{\Delta}^2-\Delta_0^2\sim m^2\Delta_0^2/I_0(m^2) \tag{31}$$

which makes all the electromagnetic terms of (24) so weak that one can neglect them, unless

$$e_B^2=\mathscr{E}_B^2 I_0(m^2)/\Delta_0^2 I_{-1}(\Delta_0^2) \tag{32}$$

where $\mathscr{E}_B^2$ can still contain logarithms of the cut-off Λ. We will now continue the study just for the case of (32), as it is the only one which might lead to a finite theory with electromagnetic interaction. For it (20) reads

$$\lambda=-\lambda_B-2/I_{-1}(m^2)[2+(4\lambda_B-\mathscr{E}_B^2)I_{-1}(m^2)]. \tag{33}$$

Consider first (26). Using its second equation and (32) the energy density reads

$$\mathscr{E}(\alpha,\bar{\Omega}^2)=\tfrac{1}{4}(\bar{\Omega}^2-m^2)^2 I_{-1}(m^2)-(\bar{\Omega}^2-m^2)\Gamma(\bar{\Omega}^2, m^2)+2\Sigma(\bar{\Omega}, m^2)$$
$$+(2\lambda_B-\tfrac{1}{2}\mathscr{E}_B^2)[\alpha-\tfrac{1}{2}(\bar{\Omega}^2-m^2)I_{-1}(m^2)+\Gamma(\bar{\Omega}^2, m^2)]^2$$
$$+m^2\alpha-\lambda_B\alpha^2+\mathrm{O}(1/\Lambda^2) \tag{34}$$

and the first of (26) is now

$$m^2-\bar{\Omega}^2+(4\lambda_B-\mathscr{E}_B^2)[\alpha-\tfrac{1}{2}(\bar{\Omega}^2-m^2)I_{-1}(m^2)+\Gamma(\bar{\Omega}^2, m^2)]+\mathrm{O}(1/\Lambda^2)=0. \tag{35}$$

There are now three phases which lead to a finite energy density and to a finite renormalised coupling constant. The first one corresponds to

$$4\lambda_B-\mathscr{E}_B^2=-\frac{2}{I_{-1}(m^2)}+\frac{c}{I_{-1}^2(m^2)}+\mathrm{O}\left(\frac{1}{I_{-1}^3(m^2)}\right) \qquad c>0 \tag{36}$$

which just satisfies the inequality of (15). On the other hand, (14) in turn allows the following values of λ_B:

$$\lambda_B=-\frac{1}{2I_{-1}(m^2)}+\frac{d}{I_{-1}^2(m^2)}+\mathrm{O}\left(\frac{1}{I_{-1}^3(m^2)}\right) \qquad d>0 \tag{37}$$

for which (25) is still positive (due to its subdominant term), or

$$\lambda_B=\frac{d}{I_{-1}(m^2)}+\mathrm{O}\left(\frac{1}{I_{-1}^2(m^2)}\right) \qquad d>-\tfrac{1}{2} \tag{38}$$

or

$$\lambda_B = d + O\left(\frac{1}{I_{-1}(m^2)}\right) \qquad d > 0. \tag{39}$$

The renormalised coupling constant is

$$\lambda = -2/c \tag{40}$$

for the first two cases and

$$\lambda = -d - 2/c \tag{41}$$

for the last one.

Equation (35) reads for the expression of (36)

$$c(\bar{\Omega}^2 - m^2) + 4(\alpha + \Gamma(\bar{\Omega}^2, m^2)) = 0 \tag{42}$$

and (34) reads

$$\mathscr{E}(\alpha; \bar{\Omega}^2) = 2\Sigma(\bar{\Omega}^2, m^2) + \tfrac{1}{8}c(\bar{\Omega}^2 - m^2)^2 + \bar{\Omega}^2\alpha - \lambda_B\alpha^2. \tag{43}$$

Equation (42) does not have a solution beyond some value of α. Then necessarily the solution of (27) applies. It leads to

$$\mathscr{E}(\alpha; 0) = -\frac{m^4}{64\pi^2} + c\frac{m^4}{8} - \lambda_B\alpha^2 - \frac{\alpha^2}{I_{-1}(m^2)}. \tag{44}$$

Notice that if $c < 1/8\pi^2$, $\mathscr{E}(0, 0) < 0$ and the assumption that (9) gives the absolute minimum is wrong, so that (10) would apply. Also $c = 1/8\pi^2$ is not within the case we are studying ($m^2 > 0$) as it corresponds to a case when (9) for $\Omega_0^2 > 0$ and (10) are equal minima; but in some neighbourhood of $\alpha = 0$ (44) lies below (43) so that really $m^2 = 0$. Sticking to our case we thus have

$$c > 1/8\pi^2. \tag{45}$$

When (39) applies (44) is unbound from below. When (37) or (38) apply the coefficient of the α^2 term is negative but infinitesimal. It goes to zero when the cut-off is removed. The regularised theory has a metastable ground state such that its lifetime becomes infinite when the cut-off is removed. It is precarious in Stevenson's sense [2] and, for $\Lambda \to \infty$, (42)-(44) read (α_c being the transition point)

$$\begin{aligned}\mathscr{E}(\alpha) = [&2\Sigma(\bar{\Omega}^2, m^2) + \tfrac{1}{8}c(\bar{\Omega}^2 - m^2)^2 + \bar{\Omega}^2\alpha]\theta(\alpha_c - \alpha) \\ &+ \tfrac{1}{8}m^4(c - 1/8\pi^2)\theta(\alpha - \alpha_c) \\ &c(\bar{\Omega}^2 - m^2) + 4(\alpha + \Gamma(\bar{\Omega}^2, m^2)) = 0 \qquad \alpha < \alpha_c.\end{aligned} \tag{46}$$

When the theory is stable, and for those values of α for which (42) and (43) apply, the energy density increases with increasing α until it reaches the solution of (44), when it becomes flat. There is no sign of spontaneous symmetry breakdown. When the theory is unstable (equation (39)) the energy density as given by (42) and (43) is non-monotonously increasing, as

$$\frac{d\mathscr{E}(\alpha; \bar{\Omega}^2)}{d\alpha} = \frac{\partial\mathscr{E}(\alpha; \bar{\Omega}^2)}{\partial\alpha} = \bar{\Omega}^2 - 2d\alpha \tag{47}$$

will become negative for small enough α (so that (42) and (43) still apply) if d is large enough. Only the unstable theory might exhibit spontaneous symmetry breakdown, but this makes no sense.

A second phase is described by

$$4\lambda_{\mathrm{B}} - \mathscr{E}_{\mathrm{B}}^2 = \mathrm{O}(1/I_{-1}^2(m^2)). \tag{48}$$

Then (35) implies

$$\bar{\Omega}^2 = m^2 + \mathrm{O}(1/I_{-1}(m^2)) \tag{49}$$

and (33) leads to $\lambda = -\lambda_{\mathrm{B}}$ so that from (34)

$$\varepsilon(\alpha) = m^2\alpha + \lambda\alpha^2 \tag{50}$$

which is the classical potential and is either trivial or unbound from below as (14) implies $\lambda \leqslant 0$. The last phase is when

$$4\lambda_{\mathrm{B}} - \mathscr{E}_{\mathrm{B}}^2 = c + \mathrm{O}(1/I_{-1}(m^2)) \qquad c > 0 \tag{51}$$

which satisfies (15). This requires

$$\lambda_{\mathrm{B}} = d + \mathrm{O}(1/I_{-1}(m^2)) \qquad d > 0 \tag{52}$$

which satisfies (14) and leads to a renormalised coupling constant

$$\lambda = -d. \tag{53}$$

Substituting (35) into (34), this last one again reads

$$\varepsilon(\alpha) = m^2\alpha - d\alpha^2 \tag{54}$$

which is unbound from below.

2.2. $\Omega_0^2 = 0, \Delta_0^2 > 0$

There are two subcases.

(i) $\quad m_{\mathrm{B}}^2 + 4\lambda_{\mathrm{B}} I_0(0) + 2e_{\mathrm{B}}^2 I_0(\Delta_0^2) = 0.$

This means that we are still with (9), but with the solution now at the border of the Ω_0^2, Δ_0^2 space. Let us assume that in some neighbourhood of $\alpha = 0$, $\Omega^2(\alpha) \neq 0$. Then the first steps of the previous analysis still apply and $m^2 = 0$. It can be seen that this case is just the limit $m^2 \to 0$ of the previous one. This then shows immediately that it cannot lead to an interacting theory, as α_{c} tends towards zero.

If $\Omega^2(\alpha) = 0$ in some neighbourhood of $\alpha = 0$ the analysis is a particular case of the next subcase, so let us consider this one first.

(ii) $\quad m_{\mathrm{B}}^2 + 4\lambda_{\mathrm{B}} I_0(0) + 2e_{\mathrm{B}}^2 I_0(\Delta_0^2) < 0.$

We are now with (10). Obviously (13) still applies. The renormalised mass is given by

$$m^2 = m_{\mathrm{B}}^2 + 4\lambda_{\mathrm{B}} I_0(0) + 2e_{\mathrm{B}}^2 I_0(\Delta_0^2) < 0 \tag{55}$$

where we have used the fact that in some neighbourhood of $\alpha = 0$, $\Omega_0^{2'} = 0$. This is because

$$\frac{\partial \mathscr{V}(0; \Omega_0^2, \Delta_0^2)}{\partial \Omega_0^2} > 0 \tag{56}$$

will, because of continuity, still be true at some neighbourhood of $\alpha = 0$ so that there $\Omega^2(\alpha) = 0$. Notice also that (55) ensures that (10) is a minimum.

The renormalised coupling constant is given by

$$\lambda = \lambda_B - e_B^2 I_{-1}(\Delta_0^2)\Delta_0^{2'} + \tfrac{1}{4}(\tfrac{3}{2} + \xi_B) I_{-1}(\Delta_0^2)(\Delta_0^{2'})^2. \tag{57}$$

Minimisation of λ leads to

$$\Delta_0^{2'} = \frac{2e_B^2}{\frac{3}{2} + \xi_B} \tag{58}$$

which then gives

$$\lambda = \lambda_B - e_B^2 \Delta_0^2 I_{-1}(\Delta_0^2)/2I_0(0). \tag{59}$$

The subtracted energy density is given by

$$\begin{aligned}\mathscr{E}(\alpha; \Omega^2, \Delta^2) = &-\frac{\Omega^4}{64\pi^2} - \frac{\Omega^2}{32\pi^2}(2m^2 - \Omega^2)\left(\ln\frac{4\Lambda^2}{\Omega^2} - 1\right)\\ &+ 2\lambda_B\left[\alpha - \frac{1}{16\pi^2}\Omega^2\left(\ln\frac{4\Lambda^2}{\Omega^2} - 1\right)\right]^2\\ &- \lambda_B\alpha^2 + m^2\alpha + 2e_B^2\left[\alpha - \frac{1}{16\pi^2}\Omega^2\left(\ln\frac{4\Lambda^2}{\Omega^2} - 1\right)\right]\\ &\times[-\tfrac{1}{2}(\Delta^2 - \Delta_0^2)I_{-1}(\Delta_0^2) + \Gamma(\Delta^2, \Delta_0^2)]\\ &+ \frac{2e_B^2 I_0(0)}{\Delta_0^2}[\tfrac{1}{4}(\Delta^2 - \Delta_0^2)^2 I_{-1}(\Delta_0^2) - (\Delta^2 - \Delta_0^2)\Gamma(\Delta^2, \Delta_0^2) + 2\Sigma(\Delta^2, \Delta_0^2)]\end{aligned} \tag{60}$$

where $\Omega^2 = \Omega^2(\alpha)$ and $\Delta^2 = \Delta^2(\alpha)$ are the functions of α which minimise (60). For fixed α and Ω^2 and Δ^2 large this equation leads to

$$\begin{aligned}\mathscr{E}(\alpha, \Omega^2, \Delta^2) \underset{\Omega^2,\Delta^2\,\text{large}}{\sim} &\frac{\Omega^4}{32\pi^2}\left(\ln\frac{4\Lambda^2}{\Omega^2} - 1\right)\left[1 + \frac{\lambda_B}{4\pi^2}\left(\ln\frac{4\Lambda^2}{\Omega^2} - 1\right)\right]\\ &+ \frac{e_B^2}{16\pi^2}\Omega^2\Delta^2\ln\frac{4\Lambda^2}{\Omega^2} I_{-1}(\Delta_0^2) + \frac{e_B^2 I_0(0)}{2\Delta_0^2} I_{-1}(\Delta_0^2)\Delta^4\end{aligned} \tag{61}$$

which requires, for stability reasons,

$$1 + \frac{\lambda_B}{4\pi^2}\left(\ln\frac{4\Lambda^2}{\Omega^2} - 1\right) > 0. \tag{62}$$

As before one finds that the only case of interest is

$$e_B^2 = \mathscr{E}_B^2 I_0(0)/\Delta_0^2 I_{-1}(\Delta_0^2) \tag{63}$$

so that

$$\lambda = \lambda_B - \tfrac{1}{2}\mathscr{E}_B^2. \tag{64}$$

The functions $\Omega^2(\alpha)$ and $\Delta^2(\alpha)$ which minimise (60) are given by

$$\begin{aligned}m^2 - \bar{\Omega}^2 + 4\lambda_B&\left[\alpha - \frac{1}{16\pi^2}\bar{\Omega}^2\left(\ln\frac{4\Lambda^2}{\bar{\Omega}^2} - 1\right)\right]\\ &+ 2e_B^2[-\tfrac{1}{2}(\bar{\Delta}^2 - \Delta_0^2)I_{-1}(\Delta_0^2) + \Gamma(\bar{\Delta}^2, \Delta_0^2)] = 0\\ &\alpha - \frac{1}{16\pi^2}\bar{\Omega}^2\left(\ln\frac{4\Lambda^2}{\bar{\Omega}^2} - 1\right) - \frac{I_0(0)}{\Delta_0^2}(\bar{\Delta}^2 - \Delta_0^2) = 0\end{aligned} \tag{65}$$

or by

$$\Omega = 0$$
$$\alpha - (I_0(0)/\Delta_0^2)(\bar{\Delta}^2 - \Delta_0^2) = 0. \qquad (66)$$

In some neighbourhood of $\alpha = 0$ necessarily this last case is operative. Then from (60)

$$\mathscr{E}(\alpha; 0, \bar{\Delta}^2) = m^2\alpha + \lambda\alpha^2. \qquad (67)$$

As (66) has a solution for all α, stability requires $\lambda > 0$. This then implies from (64) that

$$\lambda_B = d + O(1/I_{-1}(m^2)) \qquad d > 0$$
$$\mathscr{E}_B^2 < 2d. \qquad (68)$$

Beyond some value of α (65) could give a lower value of the energy density than (66). Equation (65) can be simplified to

$$m^2 - \bar{\Omega}^2 + (4\lambda_B - \mathscr{E}_B^2)\left[\alpha - \frac{1}{16\pi^2}\bar{\Omega}^2\left(\ln\frac{4\Lambda^2}{\bar{\Omega}^2} - 1\right)\right] = 0 \qquad (69)$$

which has a solution for all values of α beyond a certain value. For this solution (60) reads for large α

$$\mathscr{E}(\alpha; \bar{\Omega}^2, \bar{\Delta}^2) \sim -d\alpha^2 \qquad (70)$$

and is thus unbound.

It is clear now that the particular case of subcase (i) which we have left corresponds again to a massless theory which is the $m^2 \to 0$ limit of subcase (ii). Thus case 2 corresponds either to a free theory or to an unstable theory.

2.3. $\Omega_0^2 \geqslant 0$, $\Delta_0^2 = 0$

This corresponds to (11) and (12). For it to be a minimum

$$\partial\mathscr{V}(0; \Omega_0^2, \Delta_0^2)/\partial\Delta_0^2|_{\Delta_0^2=0} \geqslant 0 \qquad (71)$$

which from the second of (8) is seen never to be true. This case never applies.

The analysis is thus completed.

3. Conclusions

Within the Gaussian approximation of a variational approach to the energy density of scalar quantum electrodynamics we have found one phase for which there exists a stable, although precarious, ground state and the theory is interacting. It is given by (36)-(38) and it implies

$$m^2 > 0$$
$$-16\pi^2 < \lambda < 0. \qquad (72)$$

This is very much like what we know for $\lambda\phi^4$ [2]. However, the renormalised parameters of (72) now do not require λ_B infinitesimal and negative, but only infinitesimal. This makes this phase lie within the reach of functional methods which require convergence of the contributions due to large quantum fluctuations.

Notice that because beyond a certain value of α the energy density is flat the renormalised coupling constant defined there would be zero. This reminds one of asymptotic freedom, but this should be taken with a pinch of salt as this is the region where the variational parameter Ω takes a boundary value and one knows that this indicates that the variational ansatz is not the adequate one, or maybe that the effective potential is not well defined there. On the other hand, the fact that this flat energy density is an upper bound of the true one does not seem to leave much choice: if the true theory is to be stable it will have to be essentially flat and thus asymptotically free. It seems as if stability of the interacting theory requires asymptotic freedom.

Also, as for $\lambda\phi^4$, stability of the interacting theory is precarious in Stevenson's sense. Finally, again as for $\lambda\phi^4$, the renormalised mass is positive and there is no spontaneous symmetry breakdown. This last point however might well be a consequence of the variational ansatz assumed, and in particular of the fact that both components of the complex field ϕ have been taken with the same variational mass Ω. A study with a more powerful ansatz is under way.

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